

Introduction to SH Lie Algebras for Physicists

Tom Lada¹ and Jim Stasheff²

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1. INTRODUCTION

Much of point particle physics can be described in terms of Lie algebras and their representations. Closed string field theory, on the other hand, leads to a generalization of Lie algebra which arose naturally within mathematics in the study of deformations of algebraic structures (Schlessinger and Stasheff, 1985, n.d.). It also appeared in work on higher spin particles (Berends *et al.*, 1985). Representation-theoretic analogs arose in the mathematical analysis of the Batalin–Fradkin–Vilkovisky approach to constrained Hamiltonians (Stasheff, 1988).

The sh Lie algebra of closed string field theory (Saadi and Zwiebach, 1989; Kugo *et al.*, 1989; Wiesbrock, 1991, 1992a, b; Wittgen and Zwiebach, 1992; Zwiebach, 1992) is defined on the full Fock complex of the theory, with the BRST differential Q . Following Zwiebach (1992), we stipulate that the string fields B_1, B_2, \dots are elements of \mathcal{H} , the Hilbert space of the combined conformal field theory of matter and ghosts. The (n -fold) string product (bracket) for genus 0 is denoted by

$$[B_1, B_2, \dots, B_n]_0$$

It has n entries, that is, n states in \mathcal{H} . Since we will deal only with genus 0, we will omit the subscript 0 henceforth. The basic equation relating these brackets and the BRST operator is:

¹Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695-8205. E-mail address: lada@math.ncsu.edu.

²Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27599-3250. E-mail address: jds.@math.unc.edu.

$$0 = Q[B_1, \dots, B_n] + \sum_{i=1}^n \pm [B_1, \dots, QB_i, \dots, B_n] \\ + \sum \sigma(\cdot, \cdot)[B_{i_1}, \dots, B_{i_j}, [B_{i_{j+1}}, \dots, B_n]] \quad (1)$$

where the second sum is over all unshuffles (see below) and $\sigma(\cdot, \cdot)$ denotes an appropriate sign.

In their work on higher-spin particles, Berends, Burgers, and van Dam consider infinitesimal gauge transformations of the form

$$\delta_\xi \phi = \partial \xi + \sum_{n=2}^{\infty} g^n T_n(\phi, \xi)$$

where T_n is n -linear in ϕ and linear in ξ and g is a "coupling constant." Notice the unusual ϕ -dependence of the transformation. They consider the case in which the commutator of two such transformations is again of this form, which they use to define

$$[\xi_1, \xi_2] = \sum_{n=0}^{\infty} g^{n+1} C_n(\phi, \xi_1, \xi_2)$$

where C_n is n -linear in ϕ and linear in each ξ_i . For this bracket to satisfy the Jacobi identity implies a whole sequence of conditions for each order g^n . In particular, if the ξ_i and ϕ_j are fields of the same sort and we take $C_n = T_{n+1}$, then to order g^2 they find

$$\partial T_2(\phi_1, \phi_2, \phi_3) = \sum_{i=1}^3 \pm T_2(\dots, \partial \phi_i, \dots) + \sum \pm T_1(T_1(\phi_{i_1}, \phi_{i_2}), \phi_{i_3})$$

while to higher order, Burgers (1985) finds the full analogs of (1).

To see the above formulas as a generalization of those for a differential graded Lie algebra is the major goal of this paper, hopefully describing the mathematical essentials in terms accessible to *physicists*.

The concept of Lie algebra can be expressed in several different ways. The most familiar are in terms of generators and relations and in terms of a bilinear "bracket" on a vector space V satisfying the Jacobi identity. In "physical" notation, let X_a be a basis for V . The bracket $[\cdot, \cdot]$ can be specified by structure constants C_{ab}^c via the formula

$$[X_a, X_b] = C_{ab}^c X_c$$

The structure constants are skew-symmetric in the lower indices a, b .

A much more subtle description appears in the homological study of Lie algebras, but it is this description which is the most useful in homological perturbation theory and in mathematical physics in the guise of BRST operators for open algebras and in the algebra for closed string field theory. This description is implicit in the somewhat more familiar dual formulation of the Chevalley and Eilenberg (1948) cochain complex for Lie algebra cohomology: An ***n-cochain*** is a skew-symmetric *n*-linear function $\omega: V \times \dots \times V \rightarrow \mathbb{R}$ and the coboundary $d\omega$ is defined by

$$d\omega(v_1, \dots, v_{n+1}) = \sum_{i < j} (-1)^{i+j} \omega([v_i, v_j], v_1, \dots, \hat{v}_1 \dots \hat{v}_j \dots, v_{n+1})$$

where the “hatted” variables are to be omitted. *With respect to a dual basis ω^a of the dual vector space V^* , we can write d as $-1/2\omega^a\omega^b C_{ab}^c \hat{\partial}_{\omega^c}$.*

We can deal directly with the vectors rather than with the multilinear “forms” at the expense of introducing a new point of view and consideration of skew-symmetric tensors (*multivectors*).

A Lie algebra is equivalent to the following data:

A vector space V (assumed finite dimensional for simplicity of exposition).

The skew (or alternating) tensor products of V , denoted

$$\bigwedge V = \left\{ \bigwedge^n V \right\}$$

with $\bigwedge^0 V = k$, the field of scalars, typically the reals \mathbb{R} or the complex numbers \mathbb{C} . A linear map

$$d: \bigwedge V \rightarrow \bigwedge V$$

which lowers n by one and is a coderivation determined by $d|\bigwedge^2 V$ such that $d^2=0$. [That d is a **coderivation** determined by $d|\bigwedge^2 V$ means just that $d(v_i) = 0$ and

$$d(v_1 \wedge \dots \wedge v_n) = \sum_{i < j} (-1)^{i+j} d(v_i \wedge v_j) \wedge v_1 \wedge \dots \wedge \hat{v}_i \wedge \dots \wedge \hat{v}_j \wedge \dots \wedge v_n$$

For example,

$$d(X_a \wedge X_b \wedge X_c) = -C_{ab}^e X_e \wedge X_c + C_{ac}^e X_e \wedge X_b - C_{bc}^e X_e \wedge X_a$$

It may not be immediately obvious, but d restricted to \bigwedge^2 is to be interpreted as a bracket: $d(v_1 \wedge v_2) = [v_1, v_2]$ and $d^2=0$ is equivalent to the Jacobi identity.

From the point of view of the skew tensor powers of V , an **sh Lie algebra (strongly homotopy Lie algebra)** is similarly equivalent to a straightforward generalization in which d is replaced by a coderivation

$$D = d_1 + d_2 + d_3 + \dots$$

where d_i lowers n by $i-1$, in particular, $d_n(v_1 \wedge \dots \wedge v_n) \in V$.

(A further generalization in which there is a term d_0 interpreted as a fixed "background" field occurs in Zwiebach's investigation of background dependence, but for this there is no mathematical precursor, to our knowledge.)

We say that D is a coderivation to summarize the several conditions

$$d_i(v_1 \wedge \dots \wedge v_n) = \sum \pm d_i(v_{i_1} \wedge \dots \wedge v_{i_j}) \wedge v_{i_{j+1}} \wedge \dots \wedge v_{i_n}$$

where the sum is over all *unshuffles* of $\{1, \dots, n\}$, that is, all permutations that keep i_1, \dots, i_j and i_{j+1}, \dots, i_n in the same relative order. A *shuffle* of two ordered sets (decks of cards) is a permutation of the ordered union which preserves the order of each of the given subsets; an *unshuffle* reverses the process; cf. Maxwell's demon. (The serious issue of signs will be addressed below.)

Notice that the old d corresponds to d_2 , since $d(v_1 \wedge v_2) = [v_1, v_2] \in V$. On the other hand, for the new D , the component d_2 no longer is of square zero by itself and hence corresponds to a bracket which does *not* necessarily satisfy the Jacobi identity. Let us look in detail at what can happen instead.

Expand $D^2 = 0$ in its homogeneous components and set them separately equal to zero. We have then

$$0) \quad d_1^2 = 0$$

so (V, d_1) is a complex or differential (graded) module. [Usually V is itself a graded module and D is a *graded* coderivation, which implies appropriate signs in applying D to $x_1 \wedge \dots \wedge x_n$. Also, d_1 typically raises (or lowers) that "internal" degree by 1.] *Perhaps the most important examples in physics are the exterior derivative on differential forms and the BRST operator on suitable fields, whether physical or ghostly.*

$$1) \quad d_1 d_2 + d_2 d_1 = 0$$

so, with appropriate sign conventions, d_2 gives a bracket $[v_1, v_2] \in V$ for which d_1 is a derivation.

$$2) \quad d_1 d_3 + d_2 d_2 + d_3 d_1 = 0$$

or equivalently

$$d_2 d_2 = -(d_1 d_3 + d_3 d_1)$$

If we further adopt the notation

$$d_3(x_1 \wedge x_2 \wedge x_3) = [x_1, x_2, x_3]$$

then we have

$$\begin{aligned} & [[v_1, v_2], v_3] \pm [[v_1, v_3], v_2] \pm [[v_2, v_3], v_1] \\ &= -d_1[v_1, v_2, v_3] \pm [d_1 v_1, v_2, v_3] \pm [v_1, d_1 v_2, v_3] \pm [v_1, v_2, d_1 v_3] \end{aligned}$$

From this, with use of the skew-symmetry of $[\cdot, \cdot]$, we see that now the Jacobi identity holds modulo the right-hand side. *In physical language, the Jacobi identity holds modulo a BRST exact term.* In the language of homological algebra, d_3 is a chain homotopy, so we say that (V, d_2) satisfies the Jacobi identity *up to homotopy* or (V, d_1, d_2, d_3) is a **homotopy Lie algebra**. The adverb “strongly” is added to refer to the other d_i .

In physical notation, restricting d_3 to $\wedge^3 V$, we can write

$$d_3(X_a \wedge X_b \wedge X_c) = C_{abc}^e X_e$$

where C_{abc}^e is skew-symmetric in the lower indices. Similarly, we can write

$$d_1 X_a = C_a^b X_b$$

Just as the Jacobi identity can be written as a quadratic equation in the C_{ab}^c , so equation 2) can be written as a quadratic equation in the C_a^b , C_{ab}^c , and C_{abc}^e .

If we adopt the notation that $d_1 = Q$ and in general

$$d_n(x_1 \wedge \dots \wedge x_n) = [x_1, x_2, \dots, x_n]$$

then the appropriate homogeneous piece of $D^2 = 0$ is (up to sign conventions and up to some constants related to conventions on the definition of $\wedge V$) precisely the equation (1) occurring in the “nonpolynomial” version of the genus-zero closed string field algebra: The sum is not only over j , but also over all $(j, n-j)$ unshuffles. Making the correspondence precise, including the appropriate signs, requires some care, as in the next section.

In the higher spin particle algebra of Berends *et al.* (1985), variations δ_ϵ do not respect a strict bracket $[\epsilon_1, \epsilon_2]$, but rather an sh Lie structure on the space of ϵ 's. In the Batalin–Fradkin–Vilkovisky operator for constraints forming an “open” algebra with structure functions, one sees a similar structure (Stasheff, 1988).

This paper is organized as follows: After establishing our notation and conventions (especially for signs), we give the formal definition of sh Lie structure and verify the equivalence with a formulation in terms of a “nilpotent” operator on $\wedge sV$. Comparison with the physics literature calls attention to some further subtleties of signs. Then we establish the sh analog of the familiar fact that commutators in an associative algebra form a Lie algebra. We next point out the relevance of these structures to $(N + 1)$ -point functions in physics. We remark on the distinction between these structures on the cohomology level and at the underlying form level and conclude with the basic theorem of Homological Perturbation Theory relating higher-order bracket operations on cohomology to strict Lie algebra structures on forms.

2. NOTATION AND CONVENTIONS

In dealing with maps of differential graded vector spaces, it is crucial to keep careful track of signs. First, there are the appropriate signs for any graded (or *super*) context. The basic convention is that whenever two symbols, of degree m and n , respectively, are interchanged, a sign of $(-1)^{mn}$ is introduced. In particular, if σ is a permutation that is acting on a string of such symbols, $e(\sigma)$ will denote the sign that results from iteration of the basic convention, i.e., $e(\sigma) = (-1)^k$, where k is the number of interchanges of odd symbols. We also let $(-1)^\sigma$ denote the usual sign of the permutation σ . It is important to note that $e(\sigma)$ does not involve $(-1)^\sigma$ as a factor. However, if all the elements in question have degree 1, then $e(\sigma) = (-1)^\sigma$.

We always regard the symmetric group \mathcal{S}_n as acting on $V^{\otimes n}$ by

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

A map of graded vector spaces $f: V^{\otimes n} \rightarrow W$ is called

$$\text{symmetric if } f(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) = e(\sigma) f(v_1 \otimes \cdots \otimes v_n)$$

and

$$\text{skew symmetric if } f(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) = (-1)^\sigma e(\sigma) f(v_1, \dots, v_n)$$

for all $\sigma \in \mathcal{S}_n$. In terms of a basis for V , we can write f in terms of its coefficients $f^{a_1 \dots a_n}$; then symmetric or skew symmetric has the usual interpretation with respect to permutations of $a_1 \dots a_n$.

Now let $V = \{V^i\}$ be a graded vector space over a field k . Let $T^*(V)$ denote the tensor vector space generated by V , i.e., $\{V^{\otimes n}\}$. We do *not*

consider $T^*(V)$ as the tensor algebra, but rather as a *coalgebra* with the standard coalgebra structure given by the diagonal

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{j=0}^n (v_1 \otimes \cdots \otimes v_j) \otimes (v_{j+1} \otimes \cdots \otimes v_n)$$

[Here $V^{\otimes 0}$ is to be identified with k and the terms with $j=0$ or $j=n$ are of the form $1 \otimes (\dots)$ and $(\dots) \otimes 1$, respectively.] The use of coalgebras is an efficient device for some of our expressions, but it is sufficient to follow the argument in terms of ordinary tensors.

In particular, we will make use of $\wedge V$, the subspace (in fact, sub-coalgebra) of $T^*(V)$ which is fixed under the above action of the symmetric group on $V^{\otimes n}$. Although we will not need the terminology, $\wedge V$ is known as the **cofree cocommutative coalgebra** generated by V with reduced diagonal given by

$$\bar{\Delta}(u_1 \wedge \cdots \wedge u_n) = \sum_{j=1}^{n-1} \sum_{\sigma} e(\sigma) (u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(j)}) \otimes (u_{\sigma(j+1)} \wedge \cdots \wedge u_{\sigma(n)})$$

where σ runs through all $(j, n-j)$ unshuffles.

3. SH LIE STRUCTURES

Let V be a differential graded vector space with differential denoted by $l_1: V_i \rightarrow V_{i-1}$. We first recall the graded version of an ordinary Lie algebra.

Definition. A **graded Lie algebra** is a graded vector space $V = \{V_p\}$ together with a graded skew commutative bracket $[\cdot, \cdot]: V \otimes V \rightarrow V$ such that $V_p \otimes V_q \rightarrow V_{p+q}$ satisfying the graded Jacobi identity:

$$[u, [v, w]] = [[u, v], w] + (-1)^{pq} [v, [u, w]]$$

An alternative “with the grading reduced by one” (meaning $V_p \otimes V_q \rightarrow V_{p+q-1}$) occurs in the Hochschild cohomology of an associative algebra as part of the structure of a Gerstenhaber (1963) algebra, a structure which is also making an appearance in physics.

Examples have existed for a long time. Perhaps the earliest is the Schouten bracket of multivectors, although it was not identified as such until long after its introduction. Similarly, the Whitehead (1941) product of homotopy groups (which has the reduced grading) and the corresponding Samelson (1953) product were also not so identified initially.

We will also consider the differential graded vector space sV , which is

defined as isomorphic to V via $(sV)_i \approx V_{i-1}$ with $\hat{l}_1(sv) = -sl_1(v)$. The use of such a suspension operator s is implicit in the Chevalley–Eilenberg complex, but for the more complicated sh Lie structures it is best to make it explicit. For similar reasons, especially subtlety of signs, and also for comparison with the corresponding sh associative algebras, we have chosen to express this section in terms of maps l_n and \hat{l}_n rather than in terms of the d_n of the introduction.

Definition. An sh Lie structure on V is a collection of skew-symmetric linear maps $l_n: \otimes^n V \rightarrow V$ of degree $2 - n$ (for cochain complexes, and $n - 2$ for chain complexes) such that

$$\sum_{i+j=n+1} \sum_{\sigma} e(\sigma)(-1)^\sigma (-1)^{(j-1)} l_i(l_j(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(j)} \otimes v_{\sigma(j+1)} \otimes \cdots \otimes v_{\sigma(n)}) = 0 \tag{2}$$

where σ runs through all $(j, n - j)$ unshuffles (Jones, 1990).

We note that the map l_2 may be viewed as a usual (graded) Lie bracket and that when $n = 3$ and $l_3 = 0$, the definition yields the usual (graded) Jacobi identity. In general, l_3 is a homotopy between the Jacobi expression and 0, while the other l_n are higher homotopies.

It is worth calling attention to the matter of degrees, both for the forms (e.g., *ghost degree in the BRST context*) and the degree of the N -fold operations. The original mathematical formulation as above of sh Lie algebras (and their sh associative algebra predecessors) assumed that the (twofold) product or bracket had degree zero, i.e., the degree of the bracket was the sum of the degrees of the factors and that the degree of $d = d_1$ was 1 (for cohomology) or -1 (for homology). The defining equality (1) then determines the degrees of the other N -fold brackets. If the bracket should itself have a degree (for example, -1 for the Whitehead product or the Gerstenhaber bracket or 1 in some physical examples), then the degrees of the other N -fold operations would be adjusted accordingly, as would the signs in (2). In particular, as Zwiebach (1992, Section 4.1) is careful to point, in closed string field theory, the degree is given by the statistics, so the degree of the two-fold bracket is 1. As he explains in detail, a simple shift in counting degrees and appropriate changes in signs establishes the equivalence of our two sets of conventions. This is precisely the shift that relates the Whitehead (1941) product to the Samelson (1953) product.

We note that in recent physics literature, the terminology *homotopy Lie algebra* is being used to denote an sh Lie algebra as defined above together with a regrading that results from the bracket having degree ± 1 . This of course results in a different set of signs in the defining equations.

We now use the graded vector space sV ; recall $(sV)_i = V_{i-1}$. Define linear maps $\hat{l}_n: \wedge^n sV \rightarrow sV$ by

$$\begin{aligned} \hat{l}_n(sv_1 \wedge \cdots \wedge sv_n) &= \begin{cases} (-1)^{\sum_{i=1}^{n/2} |v_{2i-1}|} sl_n(v_1 \otimes \cdots \otimes v_n), & n \text{ even} \\ -(-1)^{\sum_{i=1}^{(n-1)/2} |v_{2i}|} sl_n(v_1 \otimes \cdots \otimes v_n), & n \text{ odd} \end{cases} \end{aligned} \tag{3}$$

where $|v_k| = \text{degree of } v_k$.

We may now further extend $\hat{l}_n: \wedge^k sV \rightarrow \wedge^{k-n+1} sV$ as a coderivation with respect to the usual coproduct on $\wedge^* sV$. Explicitly, we have

$$\begin{aligned} \hat{l}_n(sv_1 \wedge \cdots \wedge sv_k) &= \sum_{\sigma} e(\sigma) \hat{l}_n(sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(n)}) \wedge sv_{\sigma(n+1)} \wedge \cdots \wedge sv_{\sigma(k)} \end{aligned} \tag{4}$$

where σ runs through all $(n, k-n)$ unshuffles. Define the degree of an element in $\wedge^* sV$ by $|sv_1 \wedge \cdots \wedge sv_k| = k + \sum_{i=1}^k |v_i|$. Note that with this grading each \hat{l}_n has degree -1 . The ordering convention in (4) with the \hat{l}_n terms listed first provides a convenient way of keeping track of the signs and the unshuffles. We observe, however, that if an element of $\wedge^k sV$ has the form

$$sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(i)} \wedge sv_{\sigma(i+1)} \wedge \cdots \wedge sv_{\sigma(i+n)} \wedge \cdots \wedge sv_{\sigma(k)}$$

where the sequence $\sigma(i+1), \dots, \sigma(i+n)$ respects the original ordering of the sv_i , we then have among the summands in (4) an equivalent term of the form

$$\begin{aligned} &(-1)^{(\sum_{p=1}^i |sv_{\sigma(p)}|)(|l_n|)} \\ &\times e(\sigma) sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(i)} \wedge \hat{l}_n(sv_{\sigma(i+1)} \wedge \cdots \wedge sv_{\sigma(i+n)}) \wedge \cdots \wedge sv_{\sigma(k)} \end{aligned}$$

by use of the basic sign conventions.

In order to calculate compositions of the l_n we require the following lemma.

Lemma.

$$\begin{aligned} \hat{l}_i(\hat{l}_j(sv_1 \wedge \cdots \wedge sv_j) \wedge sv_{j+1} \wedge \cdots \wedge sv_{j+i-1}) &= (-1)^{\sum_{p=1}^{i+j-2/2} |v_{2p}|} sl_i(l_j(v_1 \otimes \cdots \otimes v_j) \otimes v_{j+1} \otimes \cdots \otimes v_{i+j-1}) \\ &\quad \text{if } i+j \text{ is even} \\ &= (-1)^{\sum_{p=1}^{i+j-1/2} |v_{2p-1}|} sl_i(l_j(v_1 \otimes \cdots \otimes v_j) \otimes v_{j+1} \otimes \cdots \otimes v_{i+j-1}) \\ &\quad \text{if } i \text{ is even and } j \text{ is odd} \\ &= (-1)^{1 + \sum_{p=1}^{i+j-1/2} |v_{2p-1}|} sl_i(l_j(v_1 \otimes \cdots \otimes v_j) \otimes v_{j+1} \otimes \cdots \otimes v_{i+j-1}) \\ &\quad \text{if } j \text{ is even and } i \text{ is odd} \end{aligned}$$

Proof. We first assume that both $i + j$ and j are even. Then, by using (3), we have

$$\begin{aligned} & \hat{l}_i(\hat{l}_j(sv_1 \wedge \cdots \wedge sv_j) \wedge sv_{j+1} \wedge \cdots \wedge sv_{i+j-1}) \\ &= (-1)^{\sum_{p=1}^{j/2} |v_{2p-1}|} (-1)^{|l_j(v_1 \otimes \cdots \otimes v_j)| + \sum_{p=1}^{i-2/2} |v_{j+2p}|} \\ & \quad \times s l_i(l_j(v_1 \otimes \cdots \otimes v_j) \otimes v_{j+1} \otimes \cdots \otimes v_{i+j-1}) \end{aligned}$$

Note that the sign in the above expression is equal to

$$(-1)^{\sum_{p=1}^{j/2} |v_{2p-1}|} (-1)^{j-2 + \sum_{p=1}^j |v_p| + \sum_{p=1}^{i-2/2} |v_{j+2p}|} = (-1)^{\sum_{p=1}^{i+j-2} |v_{2p}|}$$

which is what is claimed. The other three cases follow from a similar calculation. ■

Theorem. Let $D: \wedge^* sV \rightarrow \wedge^* sV$ be given by $D = \sum_i \hat{l}_i$. Then $D^2 = 0$.

Note. This gives a generalization of the Chevalley–Eilenberg complex for a Lie algebra. The rest of this section is devoted to a proof which amounts to checking that the signs cancel appropriately.

Proof. Evaluate $D^2 = \sum_{k=2}^{n+1} \sum_{i+j=k} \hat{l}_i \hat{l}_j$ on $sv_1 \wedge \cdots \wedge sv_n$. In fact we claim that for each $k \leq n + 1$, we have $\sum_{i+j=k} \hat{l}_i \hat{l}_j(sv_1 \wedge \cdots \wedge sv_n) = 0$. We use (4) to write

$$\begin{aligned} & \hat{l}_i \hat{l}_j(sv_1 \wedge \cdots \wedge sv_n) \\ &= \hat{l}_i \left(\sum_{\sigma} e(\sigma) \hat{l}_j(sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(j)}) \wedge sv_{\sigma(j+1)} \wedge \cdots \wedge sv_{\sigma(n)} \right) \end{aligned}$$

Now use (4) again to evaluate \hat{l}_i . At this point we note that a typical summand of the resulting expression will have the form

$$\begin{aligned} \text{(a)} \quad & e(\sigma) \hat{l}_i(\hat{l}_j(sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(j)}) \wedge sv_{\sigma(j+1)} \wedge \cdots \wedge sv_{\sigma(j+i-1)}) \\ & \wedge sv_{\sigma(j+i)} \wedge \cdots \wedge sv_{\sigma(n)}) \end{aligned}$$

or

$$\begin{aligned} \text{(b)} \quad & e(\sigma) \hat{l}_i(sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(i)}) \wedge \hat{l}_j(sv_{\sigma(i+1)} \wedge \cdots \wedge sv_{\sigma(i+j)}) \\ & \wedge sv_{\sigma(i+j+1)} \wedge \cdots \wedge sv_{\sigma(n)} \end{aligned}$$

Here σ refers to a composition of unshuffles and $e(\sigma)$ is the resulting product of their signs.

We begin with the type (a) terms. First collect all terms together that have identical last $n - j - i + 1$ entries and denote this sequence of terms by sv_Q . Then

$$\begin{aligned}
 & \sum_{i+j=k} \sum_{\sigma} e(\sigma) \hat{l}_i \hat{l}_j (sv_{\sigma(1)} \wedge \cdots \wedge sv_{\sigma(j)}) \wedge sv_{\sigma(j+1)} \wedge \cdots \wedge sv_{\sigma(j+i-1)} \wedge sv_Q \\
 &= (-1)^{\sum_{p=1}^{i+j-2/2} |v_{2p}|} s \left(\sum_{i+j=k} \sum_{\sigma} e(\sigma) (-1)^{\sigma} l_i (l_j (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(j)} \right. \\
 & \quad \left. \otimes v_{\sigma(j+1)} \otimes \cdots \otimes v_{\sigma(j+i-1)}) \right) \wedge sv_Q \tag{5}
 \end{aligned}$$

if $i + j$ is even, and

$$\begin{aligned}
 &= (-1)^{\sum_{p=1}^{i+j-2/2} |v_{2p-1}|} \\
 & \quad \times s \left(\sum_{i+j=k} \sum_{\sigma} e(\sigma) (-1)^{\sigma} (-1)^i \right. \\
 & \quad \left. \times l_i (l_j (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(j)}) \otimes v_{\sigma(j+1)} \otimes \cdots \otimes v_{\sigma(j+i-1)}) \right) \wedge sv_Q \tag{6}
 \end{aligned}$$

if $i + j$ is odd. Here σ is taken over all $(j, i - 1)$ unshuffles in the first $j + i - 1$ coordinates. Both (5) and (6) are equal to 0 by (2).

To verify this claim we use the fact that each unshuffle is the product of transpositions, and examine the effect of one transposition on the term

$$\hat{l}_i (\hat{l}_j (sv_1 \wedge \cdots \wedge sv_j) \wedge sv_{j+1} \wedge \cdots \wedge sv_{i+j-1})$$

while using the lemma to evaluate $\hat{l}_i \hat{l}_j$. In fact, the transposition that we need consider is $\sigma = (j, j + 1)$. Consequently, we have

$$\begin{aligned}
 & (-1)^{|sv_j| |sv_{j+1}|} \hat{l}_i (\hat{l}_j (sv_1 \wedge \cdots \wedge sv_{j+1}) \wedge sv_j \wedge \cdots \wedge sv_{i+j-1}) \\
 &= (-1)^{|v_j| |v_{j+1}| + 1 + |v_j| + |v_{j+1}|} \\
 & \quad \times \hat{l}_i (\hat{l}_j (sv_1 \wedge \cdots \wedge sv_{j+1}) \wedge sv_j \wedge \cdots \wedge sv_{i+j-1}) \\
 &= e(\sigma) (-1)^{\sigma} (-1)^{|v_j| |v_{j+1}|} (-1)^{\sum_{p=1}^{i+j-2/2} |v_{\sigma(2p)}|} \\
 & \quad \times sl_i (l_j (v_1 \otimes \cdots \otimes v_{\sigma(j)}) \otimes v_{\sigma(j+1)} \otimes \cdots \otimes v_{i+j-1})
 \end{aligned}$$

by the lemma when $i + j$ is even. We note that since $\sigma(j) = j + 1$ and since j and $j + 1$ have opposite parity, the exponent is

$$|v_j| + |v_{j+1}| + \sum_{p=1}^{i+j-2/2} |v_{\sigma(2p)}| = \sum_{p=1}^{i+j-2/2} |v_{2p}|$$

The case in which $i + j$ is odd follows by a similar calculation.

We now turn our attention to the terms of type (b). These terms occur in pairs with opposite signs and thus cancel each other out. They will occur

when the sequences $\sigma(1), \dots, \sigma(i)$ and $\sigma(i+1), \dots, \sigma(i+j)$ are ordered with respect to the usual ordering on the integers. We then evaluate

$$\begin{aligned} & \hat{l}_i \hat{l}_j (sv_{\sigma(1)} \wedge \dots \wedge sv_{\sigma(i)} \wedge sv_{\sigma(i+1)} \wedge \dots \wedge sv_{\sigma(i+j)} \wedge sv_Q) \\ &= (-1)^{(\sum_{p=1}^i |sv_{\sigma(p)}|) |l_j|} \\ & \quad \times \hat{l}_i (sv_{\sigma(1)} \wedge \dots \wedge sv_{\sigma(i)}) \wedge \hat{l}_j (sv_{\sigma(i+1)} \wedge \dots \wedge sv_{\sigma(i+j)}) \wedge sv_Q \\ &= (-1)^{\sum_{p=1}^i |sv_{\sigma(p)}|} \\ & \quad \times \hat{l}_i (sv_{\sigma(1)} \wedge \dots \wedge sv_{\sigma(i)}) \wedge \hat{l}_j (sv_{\sigma(i+1)} \wedge \dots \wedge sv_{\sigma(i+j)}) \wedge sv_Q \end{aligned}$$

On the other hand,

$$\begin{aligned} & \hat{l}_j \hat{l}_i (sv_{\sigma(1)} \wedge \dots \wedge sv_{\sigma(i+j)} \wedge sv_Q) \\ &= \hat{l}_j (\hat{l}_i (sv_{\sigma(1)} \wedge \dots \wedge sv_{\sigma(i)}) \wedge sv_{\sigma(i+1)} \wedge \dots \wedge sv_{\sigma(i+j)} \wedge sv_Q) \\ &= (-1)^{(\sum_{p=1}^i |sv_{\sigma(p)}|) |l_j|} (-1)^{|l_i| |l_j|} \\ & \quad \times \hat{l}_i (sv_{\sigma(1)} \wedge \dots \wedge sv_{\sigma(i)}) \wedge \hat{l}_j (sv_{\sigma(i+1)} \wedge \dots \wedge sv_{\sigma(i+j)}) \wedge sv_Q \end{aligned}$$

Since $|\hat{l}_k| = -1$, the signs are as claimed. Note that in the evaluation of \hat{l}_i and \hat{l}_j here, we were only interested in the summands listed. ■

4. COMMUTATORS IN RELATION TO THE SH ASSOCIATIVE CASE

Physicists often refer to Lie brackets as commutators because commutators of elements in an associative algebra define a bracket satisfying the axioms for a Lie algebra: $[x, y] = xy - yx$ or, in the graded case, $[x, y] = xy - (-1)^{|x||y|} yx$. There is a notion of strongly homotopy associative (sha) algebra, much older than that of sh Lie algebra (Stasheff, (1963b, 1970)); it is natural to try to put into the homotopy context the canonical construction of a Lie algebra from an associative algebra by use of commutators.

Definition. A strongly homotopy associative (sha) structure on V is a collection of linear maps $m_n: \otimes^n V \rightarrow V$ of degree $2-n$ (for cochain complexes, and $n-2$ for chain complexes) such that

$$\begin{aligned} & \sum_{i+j=n+1} \sum_k \beta(i, j, k) m_i(v_1 \otimes \dots \otimes v_{k-1} \\ & \otimes m_j(v_k \otimes \dots \otimes v_{k+j-1}) \otimes \dots \otimes v_n) = 0 \end{aligned} \tag{7}$$

where $\beta(i, j, k)$ is the sign given by the parity of $(j + 1)k + j(n + \sum_{m=1}^{k-1} |x_m|)$.

Definition. Given a differential graded vector space V with an sha structure $\{m_n\}$, the **commutator sh Lie structure** is defined on V by

$$l_n(v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma} (-1)^{\sigma} e(\sigma) m_n(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})$$

where the sum is taken over all permutations $\sigma \in S_n$.

When $n = 2$,

$$l_2(v_1 \otimes v_2) = m_2(v_1 \otimes v_2) - (-1)^{|v_1||v_2|} m_2(v_2 \otimes v_1)$$

is the graded commutator. The possible lack of associativity of m_2 will in general prevent the Jacobi expression

$$l_2(l_2(v_1 \otimes v_2) \otimes v_3) - (-1)^{|v_2||v_3|} l_2(l_2(v_1 \otimes v_3) \otimes v_2) + (-1)^{|v_1|(|v_2| + |v_3|)} l_2(l_2(v_2 \otimes v_3) \otimes v_1) \tag{8}$$

from being equal to zero. However, with

$$l_3(v_1 \otimes v_2 \otimes v_3) = \sum_{\sigma \in S_3} (-1)^{\sigma} e(\sigma) m_3(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)})$$

one may check that (8) is equal to $l_1 l_3 + l_3 l_1$ where l_1 is defined to be m_1 . These details, as well as those required for 4-tensors, are made explicit by Jones (1990). The general verification that the above-defined “commutator” l_n ’s satisfy the defining equation for an sh Lie structure will appear elsewhere.

5. $(N + 1)$ -POINT FUNCTIONS

As a generalization of Lie algebras, sh Lie algebras appear in closed string field theory as symmetries or gauge transformations. The corresponding Lagrangians consist of (sums of) $(N + 1)$ -point functions for all N (Saadi and Zwiebach, 1989; Kugo *et al.*, 1989; Kaku, 1988a–d; Wiesbrock, 1991, 1992a, b; Zwiebach, 1992; Witten and Zwiebach, 1992). They can be regarded as being formed from the N -fold brackets $[x_1, x_2, \dots, x_N]$, by evaluation with a dual field via an inner product, or the $(N + 1)$ -point functions can be described directly. The latter is more appropriate for Kontsevich’s (1992) new invariants, but we will leave it to him to present those details. We adopt Dirac’s bra-ket notation and so write $|x\rangle$ instead of just x . We then write $\langle x|$ for elements of the dual space. In terms of a basis x_{α} or rather $|x_{\alpha}\rangle$, we have a dual basis $\langle x_{\alpha}|$ with

$$\langle x_{\alpha} | x_{\beta} \rangle = \delta_{\alpha\beta}$$

In terms of the N -fold bracket, we then define

$$\{y_0 y_1 \cdots y_N\} = \langle y_0 | [y_1, y_2, \dots, y_N] \rangle$$

Zwiebach streamlines the machinery in Kugo *et al.* (1989), giving the classical action in closed string field theory, the gauge transformations, and showing the invariance of the action. The classical string action is simply given by

$$S(\Psi) = \frac{1}{2} \langle \Psi, Q\Psi \rangle + \sum_{n=3}^{\infty} \frac{\kappa^{n-2}}{n!} \{\Psi \cdots \Psi\}$$

The expression $\{\Psi \cdots \Psi\}$ contains n terms and will be abbreviated $\{\Psi^n\}$, and similarly for $[\Psi^n, \Lambda]$ below.

The field equations follow from the classical action by simple variation:

$$\delta S = \sum_{n=2}^{\infty} \frac{\kappa^{n-2}}{n!} n \{\delta\Psi, \Psi^{n-1}\}$$

The gauge transformations of the theory are given by

$$\delta_{\Lambda} |\Psi\rangle = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} [\Psi^n, \Lambda]$$

Notice that *all* the terms of higher order are necessary for these to be consistent.

Similarly, in the sha context (cf. open string field theory as remarked below), one can define $(N+1)$ -point functions using the structure maps m_N . A particular example of such a structure which has quite recently appeared in the physics literature (Witten, 1992) can be expressed in terms of Massey products: Let M be a compact oriented manifold of dimension m and H its deRham cohomology. Poincaré duality gives a graded inner product on H and a pairing $\langle \cdot | \cdot \rangle$ between homology and cohomology. Massey products are determined by N -linear maps $H \otimes \cdots \otimes H \rightarrow H$ and $(N+1)$ -point functions then follow. Because the N -linear map has degree $2-N$ and the inner product is nonzero only for classes whose degrees sum to m , the only case in which all N -point functions $\langle yy \cdots y \rangle$ have the same degree (*and hence can appear simultaneously in a single Lagrangian*) occurs when the degree of y is 1 and $m=2$. (A threefold Massey product in the complement of the Borromean rings detects the nontriviality of that link.) *With the grading conventions of closed string field theory, as in Zwiebach (1992), the nonpolynomial Lagrangian has all fields Ψ of ghost degree 2.*

6. FIELDS: FORMS VERSUS COHOMOLOGY CLASSES

Recently the point of view of “cohomological physics” has become fairly common. Theories, both Lagrangian and Hamiltonian, are described initially in terms of a differential graded vector space or module (the differential is often referred to as a *BRST operator*), but the physical states are often cohomology classes, represented by closed “forms.” The structures we have been discussing, whether sh Lie or sh associative, may occur at either level (Retakh, 1977, n.d.; Barnes and Lambe, 1991; Stasheff, 1963b, 1970). For example, closed string field theory (Saadi and Zweibach, 1989; Kugo *et al.*, 1989; Zweibach, 1992) exhibits the structure of an sh Lie algebra initially at the form level, but there is an induced structure on cohomology (where $d_1 = 0$, of course). With $d_1 = 0$, the bracket $[\cdot, \cdot]$ satisfies the Jacobi identity strictly, but this does *not* imply that the higher-order brackets must be zero. Witten and Zwiebach (1992) describe the cohomology structure only, but there is one implicit at the form level as well, which Zwiebach (1992) works out very carefully.

In the open string field theory of Hata *et al.* (1986, 1987) the structure is **homotopy associative** (the associating homotopy m_3 corresponds to the trilinear operation $(\circ \circ)$ on three open string fields), but the higher-order homotopies are zero. That is, Hata *et al.* define a convolution product of string field $\Phi * \Psi$ and establish the relation

$$\begin{aligned} (\Phi * \Psi) * \Lambda - \Phi * (\Psi * \Lambda) = & \pm Q(\Phi \circ \Psi \circ \Lambda) \pm (Q\Phi \circ \Psi \circ \Lambda) \\ & \pm (\Phi * Q\Psi \circ \Lambda) \pm (\Phi \circ \Psi \circ Q\Lambda) \end{aligned}$$

but, proceeding to four fields, they find (with appropriate signs)

$$\begin{aligned} \Phi * (\Psi \circ \Lambda \circ \Sigma) \pm (\Phi \circ \Psi \circ \Lambda) * \Sigma \pm (\Phi * \Psi) \circ \Lambda \circ \Sigma \\ \pm \Phi \circ (\Psi * \Lambda) \circ \Sigma \pm \Psi \circ \Psi \circ (\Lambda * \Sigma) \end{aligned}$$

vanishes, rather than being equal to a BRST exact term; cf. (7). However, there are likely to be nontrivial higher order N -linear operations on the cohomology.

A reasonably general result that derives higher-order operations on cohomology even when the underlying differential graded algebra is strictly associative or strictly Lie is given by Homological Perturbation Theory (**HPT**):

Theorem. Let (V, d_1) be a differential graded vector space of the same homotopy type as a differential graded algebra (either associative or Lie) (A, d_A) . A specific homotopy equivalence induces the structure of an sh (respectively associative or Lie) algebra on V .

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